Fractals Chapter 4

Term coined by Benoit Mandelbrot in the 1960's to describe roughness.

Zoom in on function and it becomes a line. Not true in real world (e.g. coast of england).

fractal-candy-3d, Mandelbrot NOVA (cellphone antenna), wiki, tweet:

- complicated structure at a range of length scales (e.g. stock prices, clouds)
- self-similarity (e.g. broccoli)
- non-integer or 'fractal' dimension

4.1 Middle-thirds Cantor Set:

$K_0 = [0,1]$
$K_1 = [0,1/3] \cup [2/3,1]$
$K_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$
$\cdots = K_\infty$

1. What is the length of K_∞?
2. Does K_∞ contain any intervals?
3. Is K_∞ simply the endpoints of the intervals removed at each stage? (vote) Note that the framing "endpoints in K_∞" is confusing, as there are an uncountable number of points.

Well, $K_\infty \subseteq K_n$ for each n, and K_n consists of 2^n intervals, each of length 3^{-n}.

So the total length is $(\frac{2}{3})^n$ which goes to zero as $n \to \infty$, so K_∞ has length 0.

A set is countable if it can be put into a 1-1 correspondence with $\mathbb{N}$. A set is uncountable if it is not countable.

Deck of cards? No, finite. Not a correspondence.

Integers? Rationals in [0,1]? Winding argument. Why can't I simply count down the first column?

$\frac{1}{2}$

$\frac{1}{3}$ $\frac{2}{3}$

$\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$

Its ok that 1/2, for example, is counted many times, since a subset of a countable set is countable.

A countable collection of countable sets is also countable (how?), so the set of all rational numbers is countable.

A set S has measure zero if it can be covered by a countable number of intervals whose total length is arbitrarily small.

ex) $\{1,2,3,4,5\}, \mathbb{N}$ rationals, K_∞ are measure zero

non-ex) irrational numbers in any interval, reals in any interval. $[0,1]$ is measure 1.

Theorem: The Middle-thirds Cantor Set K_∞ consists of all numbers in the interval $[0,1]$ that can be expressed in base 3 (ternary) using only the digits 0 and 2. Proof to come. Consider

$$\begin{aligned} r &= 0.\overline{02}_3 \in K_\infty \\ &= 0\cdot 3^{-1} + 2\cdot 3^{-2} + 0\cdot 3^{-3} + 2\cdot 3^{-4} + \dots \\ &= \frac{2}{9}(1 + 3^{-2} + 3^{-4} + \dots) = \frac{2}{9}\left(\frac{1}{1-\frac{1}{9}}\right) = 1/4 \end{aligned}$$

1/4 is in the bottom third (not removed at the first step), in the top third of the bottom third, in the bottom third of that, in the top third of that, and so on... alternating between top and bottom thirds.

Since it is never in one of the middle thirds, it is never removed, and yet it is also not one of the endpoints of any middle third.

ex) the set of left endpoints of Middle thirds Cantor Set intervals is countable.

The subset of K_∞ with a finite number of repeating ternary digits (e.g. ending in $\overline{0}, \overline{2}$, or $\overline{202}$) is countable.

However, K_∞ is uncountable. Why? Lets try and list the numbers in K_∞.

$\mathbb{N}$	ternary # in K_∞
1	$r_1 = 0.a_{11}a_{12}a_{13}\ldots$
2	$r_2 = 0.a_{21}a_{22}a_{23}\ldots$
3	$r_3 = 0.a_{31}a_{32}a_{33}\ldots$
...	...
n	$r_n = 0.a_{n1}\ldots a_{nn}\ldots$
...	...

where $a_{ij} \in \{0,2\}$. Is this all of the numbers in K_∞?

Define $r = 0.b_1b_2b_3\ldots$ such that $b_1 = 0$ if $a_{11} = 2$ and $b_1 = 2$ if $a_{11} = 0$.

Do this for all b_i including $b_n \neq a_{nn}$. Then $r \in K_\infty$ but has no corresponding integer in $\mathbb{N}$.

If our list contained only rationals, this number r would be an irrational and not in the list.

Therefore K_∞ is uncountable.

The same is true of $[0,1]$ and the irrationals. This is why the probability of choosing a rational at random in $[0,1]$ is zero.

K_∞ is a **pathological** example. An uncountable set with measure 0.

Make it to HERE on day 1 back from spring break.

4.2 Probabilistic Construction of Fractals: Start with any point in the interval $[0,1]$ and flip a coin. If heads, move the point two-thirds of the way to 1. If tails, move the point two-thirds of the way to 0.

This is the same process as iterating the maps $f_1(x) = x/3$ and $f_2(x) = (2+x)/3$ with equal probability. Show matlab script `probability_cantor.m` K_∞ is the attractor for this process.

After k flips of the coin, the randomly generated orbit must lie within $\frac{1}{3^k 6}$ of a point in K_∞. Why?

Furthermore, K_∞ is invariant under this operation, i.e. if you move a point in K_∞ two-thirds of the way to either 0 or 1, the resulting point is still in K_∞. Why?

f_1 moves ternary point to the left. f_2 moves ternary point to the left and then places 2 in the thirds digit.

Sets with repeated patterns on smaller scales are called self-similar. Show figures 4.7 (an orbit) and 4.8 (basins).

4.4 Fractal Basin Boundaries: Consider a square $R \subseteq \mathbb{R}^2$ whose image under a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an S (i.e. stretch east-west by a factor of roughly 10 and bend).

Points outside R are attracted to either sink.

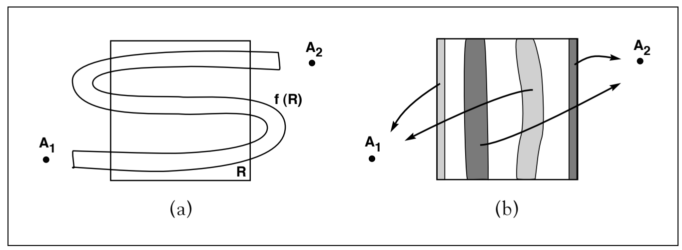

Figure 4.9 Construction of a fractal basin boundary.
(a) The image of the rectangle R is an S-shaped strip. Points that map outside and to the left of R are attracted to the sink A_1, and points that map outside and to the right of R are attracted to the sink A_2. (b) The shaded regions are mapped out of the rectangle in one iteration. Each of the three remaining vertical strips will lose 4 shaded substrips on the next iteration, and so on. The points remaining inside forever form a Cantor set.

The three remaining vertical white strips in R stay in R for one iterate of f.

Pick one of these vertical strips (far right is top horizontal strip). On the next iteration it is stretched into an S and loses two substrips to A_1, two to A_2. Continuing, the subset of R remaining inside R after n iterates consists of 3^n vertical strips whose width goes to zero as $n \to \infty$.

Therefore there is a Cantor Set of vertical curves whose images remain inside R for all n. Each curve stretches the vertical length of R, and each point in the set has nearby points, as close as you want, that go to A_1, same for A_2.

The Cantor Set is the boundary between the basins of attraction of the sinks A_1 and A_2. Think regimes of the climate.
tweet: `https://youtu.be/ovJcsL7vyrk`

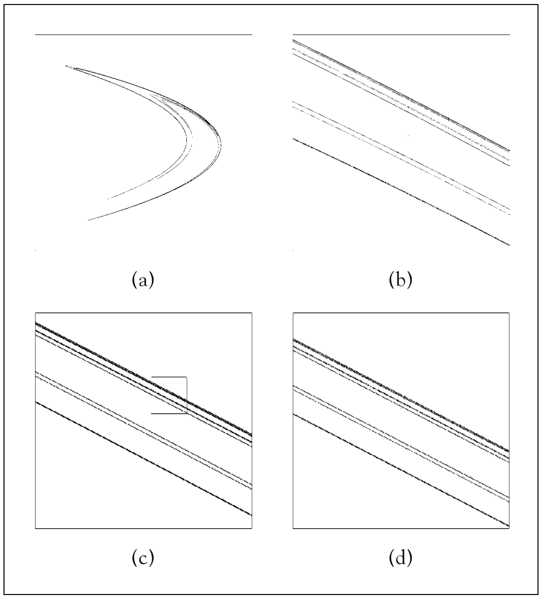

Figure 4.7 Self-similarity of the Hénon attractor.
(a) An attracting orbit of (4.1). Parts (b),(c),(d) are successive magnifications, showing the striated structure repeated on smaller and smaller scales. This sequence is zooming in on a fixed point. (a) $[-2.5, 2.5] \times [-2.5, 2.5]$. (b) $[0.78, 0.94] \times [0.78, 0.94]$. (c) $[0.865, 0.895] \times [0.865, 0.895]$. (d) $[0.881, 0.886] \times [0.881, 0.886]$.

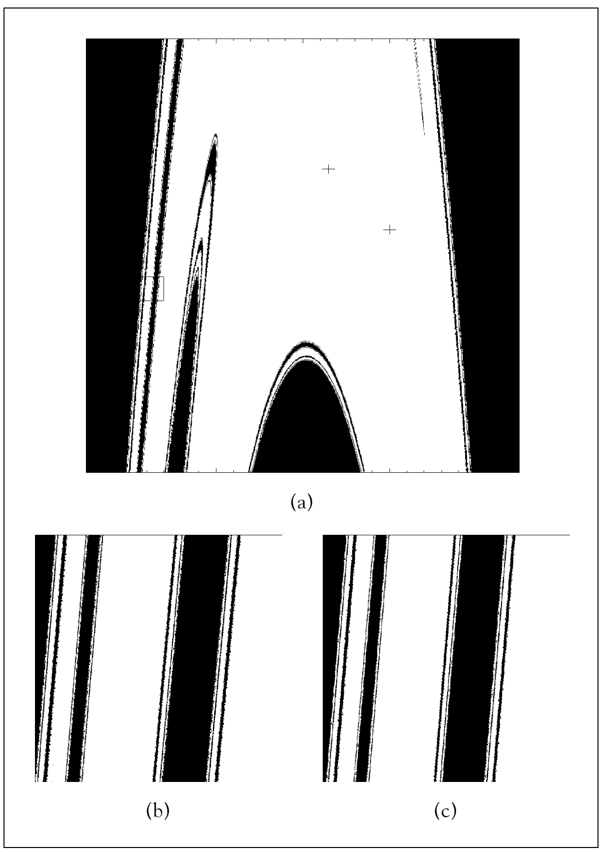

Figure 4.8 Self-similarity of the Hénon basin.
The points in white are attracted to the period-two attractor $\{(1, 0.3), (0.3, 1)\}$ of (4.2), marked with crosses. The points in black are attracted to infinity with iteration. (a) The region $[-2.5, 2.5] \times [-2.5, 2.5]$. (b) The subregion $[-1.88, -1.6] \times [-0.52, -0.24]$, which is the box in part (a). (c) The subregion $[-1.88, -1.86] \times [-0.52, -0.5]$, the box in the lower left corner of part (b).

Julia Sets

Consider the quadratic map $P_c(z) = z^2 + c$ where $z = x + yi$ and $c = a + bi$ are complex numbers ($i = \sqrt{-1}$). Evaluating the map, we find

$$P_c(z) = (x+yi)^2 + a + bi = x^2 - y^2 + a + (2xy + b)i$$

If $c = 0$, the map $P_0(z) = z^2$ has an attracting fixed point at the origin with basin $\{z : |z| < 1\}$ (i.e. the interior of the unit disk).

Points on the unit disk have their angle doubled, but stay on the disk. Points outside the disk are in the basin of infinity.

For nonzero c, the equation $z^2 + c = z$ has roots, implying P_c has fixed points.

Since P_c has only one critical point ($P_c'(z) = 0$), namely the origin, it can have at most one sink (by Schwarzian thm).

Lets try $c = -i$, i.e. $a = 0, b = -1$. Our map is $P_{-i}(z) = z^2 - i$.

Iterating the origin, i.e. $x = 0, y = 0$, write these out and draw points in complex plane

$$P_{-i}(0) = -i$$
$$P_{-i}(-i) = -1 - i$$
$$P_{-i}(-1-i) = i$$
$$P_{-i}(i) = -1 - i$$

The orbit $\{-1-i, i\}$ is a period-2 source to which the origin is eventually periodic.

Now, every attracting orbit for a polynomial map P attracts at least one critical point of P (Fatou).

Since $P_c(z)$ has only one critical point, and $P_{-i}(0)$ is EP to a repelling period-2 orbit, there is no sink! i.e. P_{-1} has no attractor.

For other values of c, some initial points will remain bounded under P_c, others will diverge to infinity.

Try these values on the computer using `julia_gui.m`

- $c = 0,\ c = -i$
- $c = -0.17 + 0.78i$, period-3 sink
- $c = 0.38 + 0.32i$, period-5 sink
- $c = 0.32 + 0.043i$, period-11 sink

The boundary between bounded and unbounded points is the Julia Set J of P_c (after Gaston Julia, French Mathematician). The boundary points are bounded (black), but not in the basin of the sink to which all other black points are AP.

The filled Julia Set is given by the set J and the points in its interior. For disconnected J, the filled Julia Set is simply J.

- J is invariant, if $z_0 \in J$, so is $z_1 = P_c(z_0), z_2, z_3, \ldots$
- orbits in J are *either* periodic sources, ep to periodic sources, or chaotic
- all unstable periodic points of P_c are in J
- J is *either* totally connected or totally disconnected, i.e. intervals or dust
- J is connected iff the orbit of the origin is bounded
- J repels orbits (finding it can be tricky, HW5)

http://universefactory.net/test/julia/

The Mandelbrot Set M is then defined to contain all values c such that the origin is *not* in the basin of infinity for the map $P_c(z) = z^2 + c$.

So $c = 0$ and $c = -i$ are in M. We can give these points a color by plotting $c = a + bi$ as (a, b) Typically, points in M are colored black, points not in M are given a color based on their rate of divergence towards infinity.

Show `mandelbrot_points.m`, Fractal Domains application, mandelbrot-zoom-hires.avi, tweet http://www.youtube.com/watch?v=Ehwy4Gq27uY `http://acko.net/blog/how-to-fold-a-julia-fractal/` `https://youtu.be/4LQvjSf6SSw`

- $c = -0.17 + 0.78i$, period-3 sink lobe at 12 o'clock
- $c = 0.38 + 0.32i$, period-5 sink lobe at 2 o'clock

Show color figure of riddled basin.

The union of the three lines contains the attracting sets for the map. Figure shows basins of second iterate of the map (6 attractors, basins are colored, basin of infinity is black).

The basins are riddled in the that any disk in the colored region contains points whose orbits move to different attractors.

Infinite accuracy would be required to predict the behavior of any point. The entire basin is boundary.

4.5 Fractal Dimension:

We'll start with a poorly motivated question. Other measures of dimension will have a more physically satisfying foundation, they came first.

Imagine laying a grid of equal spacing on top of a set. How does the number of boxes necessary to cover the set vary as the grid size is decreased?

ex) For the interval $[0,1]$, we need n boxes (subintervals) of size $\frac{1}{n}$. Draw.

In general, the number of boxes of size ε (small), $N(\varepsilon)$, required to cover an interval is less than or equal to $\frac{C}{\varepsilon}$ where C is a constant analogous to area.

In other words, $N(\varepsilon)$ scales as $\frac{1}{\varepsilon}$.

ex) Consider the square $[0,2]\times[0,2]$ in $\mathbb{R}^2$

$N(\varepsilon)$	$\log_2(N)$	ε	$\log_2(1/\varepsilon)$
1	0	2^1	-1
4	2	2^0	0
16	4	2^{-1}	1
64	6	2^{-2}	2
...	...	...	...
$\frac{4}{\varepsilon^2}$	$2(1+\log_2(1/\varepsilon))$	ε	$\log_2(1/\varepsilon)$

Plot $\log_2\left(N(\varepsilon)\right)$ as a function of $\log_2\left(\frac{1}{\varepsilon}\right)$, the square is covered by $\frac{4}{\varepsilon^2}$ boxes of side length ε.

More generally, a set $S\subseteq\mathbb{R}^m$ is said to be d-dimensional if it can be covered by a $N(\varepsilon)=\frac{C}{\varepsilon^d}$ boxes of side length ε in the limit as $\varepsilon\to 0$.

C can be as large as is needed, provided the ε^{-d} scaling holds as $\varepsilon\to 0$. d need not be an integer.

$$d=\frac{\ln\left(N(\varepsilon)\right)-\ln C}{\ln\frac{1}{\varepsilon}}$$

In the limit as $\varepsilon\to 0$, $\ln C$ becomes negligible. A bounded set S in $\mathbb{R}^n$ has box-counting dimension

$$boxdim(S)=\lim_{\varepsilon\to 0}\frac{\ln\left(N(\varepsilon)\right)}{\ln\frac{1}{\varepsilon}}$$

if the limit exists. To make the definition easier to use, we'll make three simplifications.

1. the limit as $\varepsilon\to 0$ need only be checked at discrete box sizes, provided sequence goes to 0.
2. boxes need not sit on a grid, they can be moved around. e.g. we could simply take the minimum number of boxes of size ε required to cover the set, arranged anyway we please.
3. sets other than boxes (discs, triangles, etc.) will be fine

Tweet these:

`http://www.ams.org/journals/notices/201209/rtx120901208p.pdf`

`http://www.ted.com/talks/kevin_slavin_how_algorithms_shape_our_world`

`http://www.wired.com/wiredscience/2012/01/the-fractal-dimension-of-zip-codes`

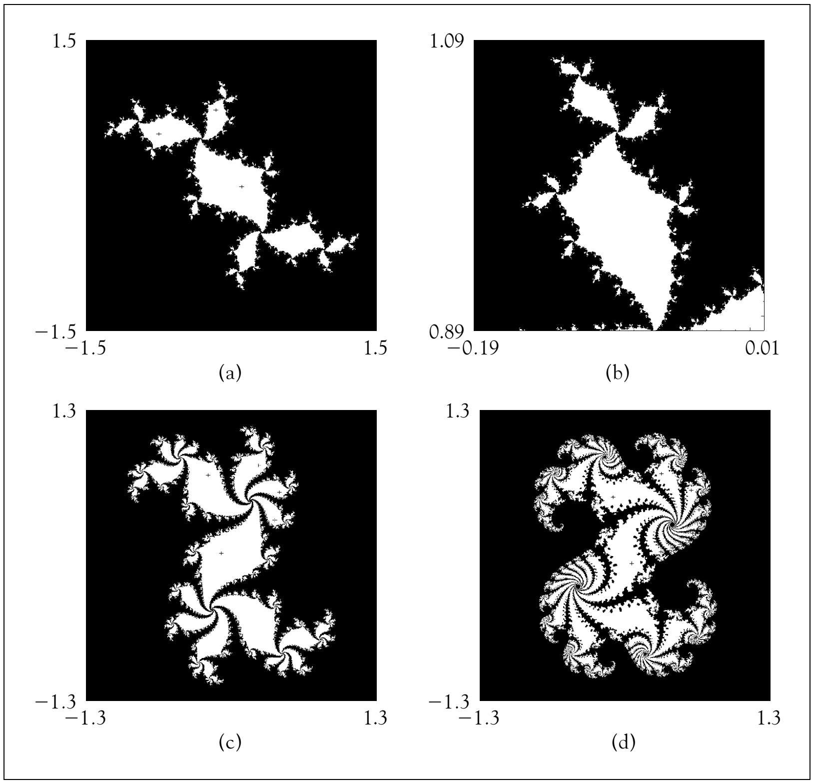

Figure 4.11 Julia sets for the map $f(z) = z^2 + c$.
(a) The constant is set at $c = -0.17 + 0.78i$. The white points are the basin of a period-three sink, marked with crosses, while the black points are the basin of infinity. The fractal basin boundary between black and white is the Julia set. (b) The uppermost rabbit of (a) is magnified by a factor of 15. (c) $c = 0.38 + 0.32i$. The white points are the basin of a period-five sink, marked with crosses. (d) $c = 0.32 + 0.043i$. The white points are the basin of a period-11 sink.

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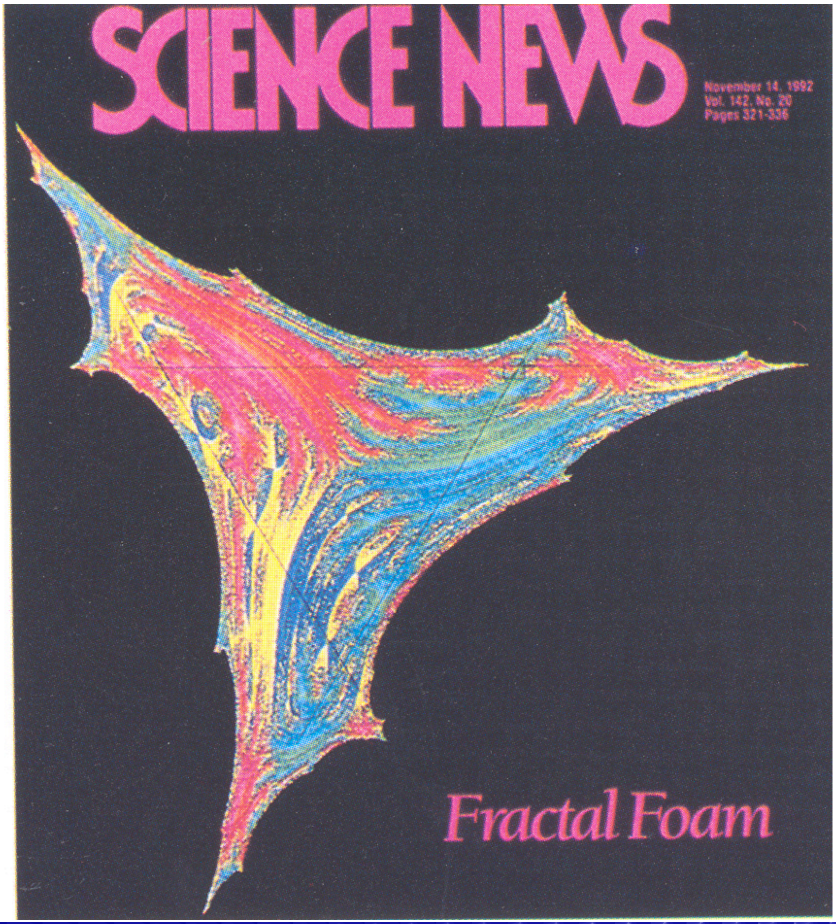
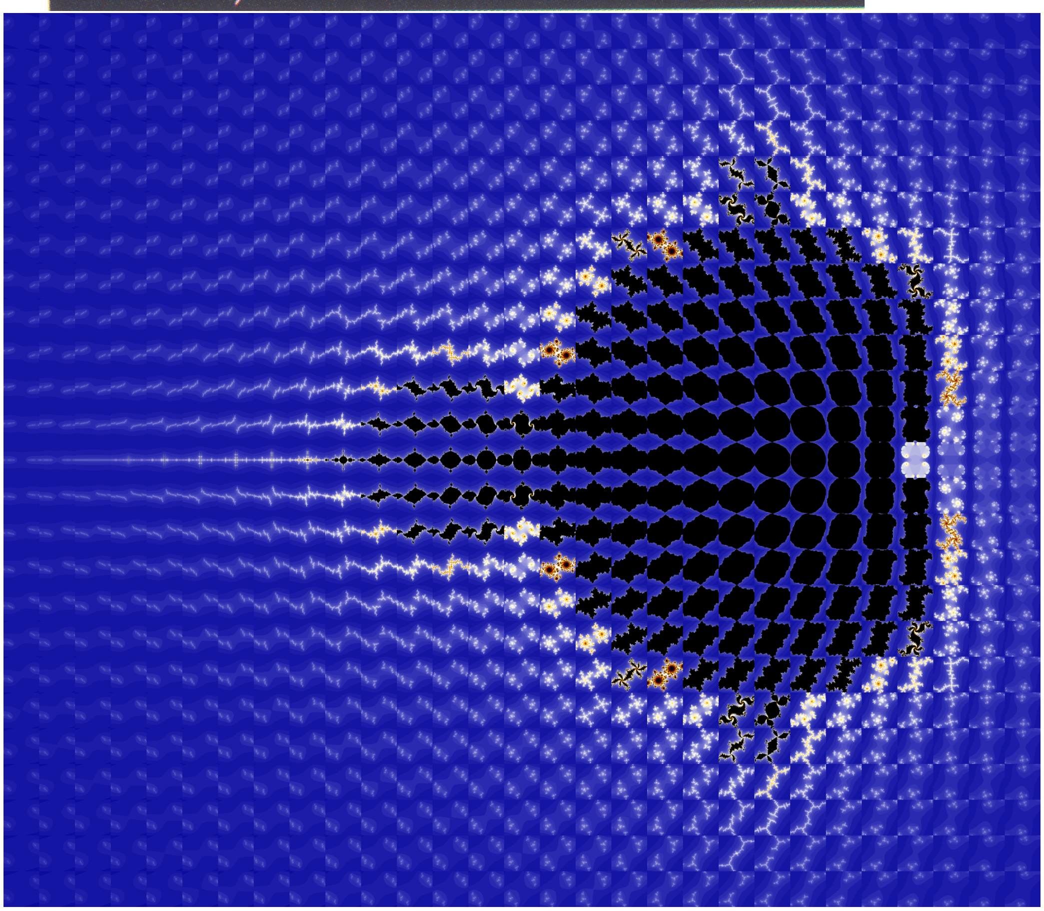

SCIENCE NEWS
November 14, 1992
Vol. 142, No. 20
Pages 321-336
Fractal Foam



4.6 Computing the Box-Counting Dimension:

ex) Middle-thirds Cantor Set, recall that $K_\infty \subseteq K_n$ where K_n consists of 2^n intervals of length 3^{-n}.

We'll use these intervals as our boxes, i.e. we'll cover K_n with 2^n boxes of size 3^{-n}. Then

$$boxdim(K_\infty) = \lim_{n\to\infty} \frac{\ln(2^n)}{\ln(3^n)} = \frac{\ln 2}{\ln 3} \approx 0.63092...$$

ex) Henon map attractor (which we have seen to be self-similar), show figures 4.13, 4.15.

Boxes N containing piece of attractor	$\log_2(N)$	Box side length ε
76	≈ 6.3	2^{-2}
177	≈ 7.5	2^{-3}
433	≈ 8.8	2^{-4}
1037	≈ 10	2^{-5}
2467	≈ 11.2	2^{-6}
5763	≈ 12.5	2^{-7}

Now again lets plot $\log_2\left(N(\varepsilon)\right)$ vs $\log_2\left(\frac{1}{\varepsilon}\right)$ and find the slope of the line. On the computer, this is the best we can do b/c $N(\varepsilon)$ becomes less reliable as $\varepsilon \to 0$.

Least squares is generally the best way to do this, mention 237.

$$boxdim(S) = \lim_{\varepsilon\to 0} \frac{\ln\left(N(\varepsilon)\right)}{\ln\frac{1}{\varepsilon}} \approx 1.27$$

Note that the slope would be the same in any base logarithm.

In $\mathbb{R}^n$ there are ε^{-n} boxes in the unit cube. ex) in 10 dimensions using boxes of size $\varepsilon = 0.01$, there are 10^{20} boxes to check. So we may need another method.

Theorem Let A be a bounded subset of $\mathbb{R}^m$ with $boxdim(A) = d < m$. Then A is a measure zero set.

- a point has measure 0 in $\mathbb{R}$
- a line has measure 0 in $\mathbb{R}^2$
- a disk has measure 0 in $\mathbb{R}^3$

Proof: We know

$$d = \lim_{\varepsilon\to 0} \frac{\ln\left(N(\varepsilon)\right)}{\ln\frac{1}{\varepsilon}} = \lim_{\varepsilon\to 0} -\frac{\ln\left(N(\varepsilon)\right)}{\ln\varepsilon}$$

where $m - d > 0$, so

$$\lim_{\varepsilon\to 0} \frac{m\ln\varepsilon + \ln\left(N(\varepsilon)\right)}{\ln\varepsilon} > 0$$

$$\lim_{\varepsilon\to 0} \frac{\ln\left(\varepsilon^m N(\varepsilon)\right)}{\ln\varepsilon} > 0$$

and since

$$\lim_{\varepsilon\to 0} \ln\varepsilon = -\infty$$

we know

$$\lim_{\varepsilon\to 0} \ln\left(\varepsilon^m N(\varepsilon)\right) = -\infty$$

also b/c anything else would imply $m - d \leq 0$.

So $\varepsilon^m N(\varepsilon) \to 0$ as $\varepsilon \to 0$. In other words, there exist ε-boxes covering A whose total volume $\varepsilon^m N(\varepsilon)$ can be made as small as we want.

Therefore A has measure zero. Other examples:

- the rationals have measure 0
- K_∞ has measure 0 (but is uncountable)
- the irrationals in $[0,1]$ have measure 1

`https://www.youtube.com/watch?v=gB9n2gHsHN4`

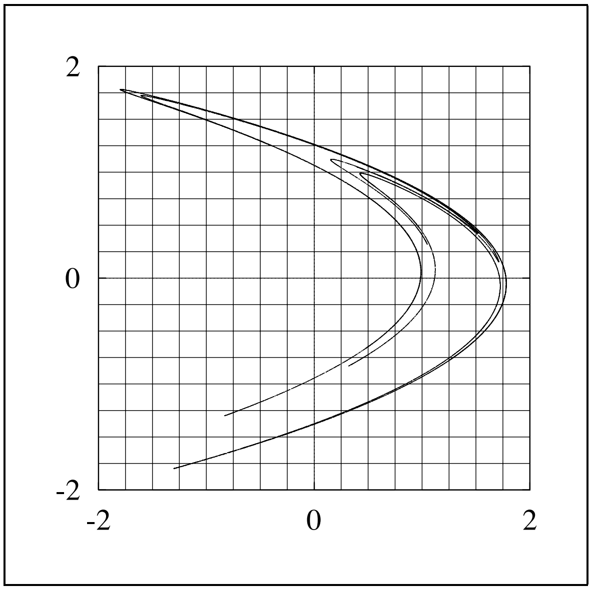


Figure 4.13 Grid of boxes for dimension measurement.
The Hénon attractor of Example 4.10 is shown beneath a grid of boxes with side-length $\epsilon = 1/4$. Of the 256 boxes shown, 76 contain a piece of the attractor.

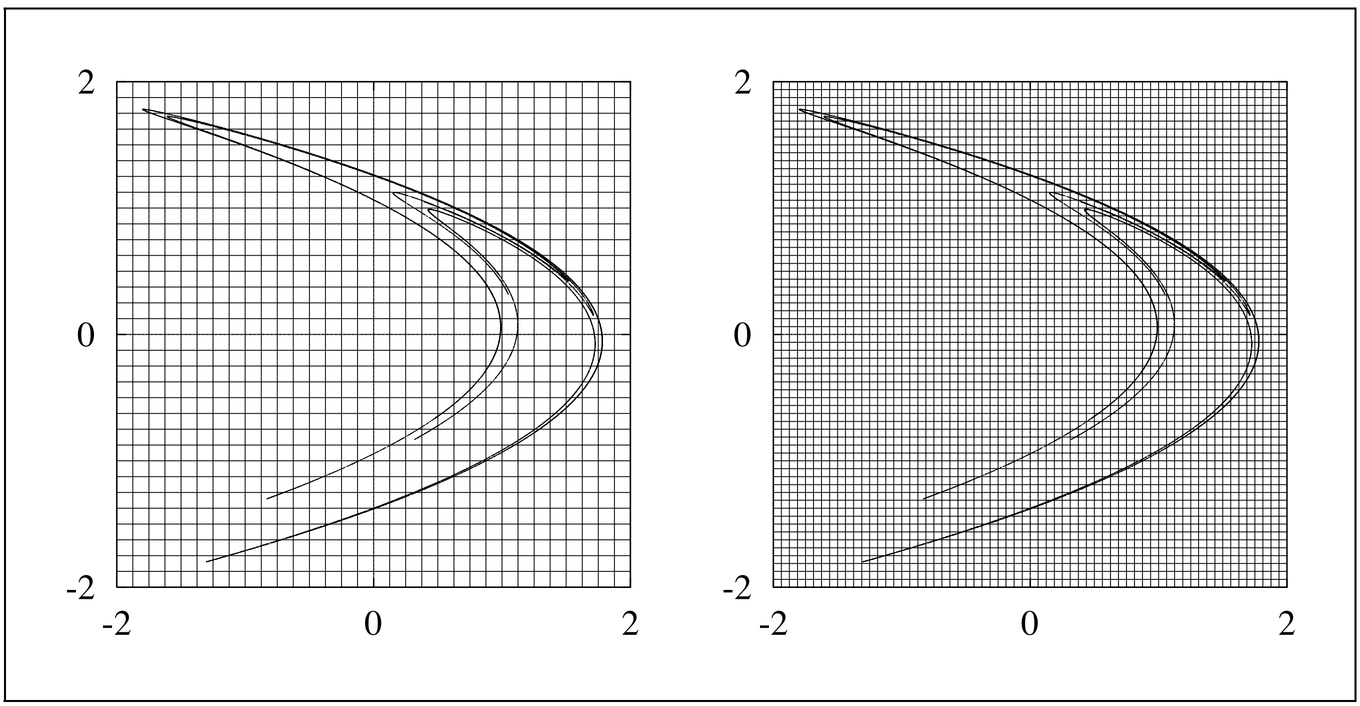


Figure 4.15 Finding the box-counting dimension of the Hénon attractor.
Two grids are shown, with gridsize $\epsilon = 1/8$ and $1/16$ respectively.

If you need to skip a lecture to get back on track for assignment 9, this is the one.

4.7 Correlation Dimension: Easier to compute, defined for an orbit rather than a set. It won't require boxes.

Let $S = \{\vec{v}_0, \vec{v}_1, \cdots\}$ be an orbit of the map $\vec{f}$ in $\mathbb{R}^n$, and S_N denote the first N points on the orbit. For each $r > 0$, define the correlation function $C(r)$ to be the proportion of pairs of orbit points within r units of one another, i.e. $C(r) =$

$$\lim_{N\to\infty} \frac{\#\{pairs\{\vec{w}_1, \vec{w}_2\} : \vec{w}_1, \vec{w}_2 \in S_N, |\vec{w}_1 - \vec{w}_2| < r\}}{\#\{pairs\{\vec{w}_1, \vec{w}_2\} : \vec{w}_1, \vec{w}_2 \in S_N\}}$$

Total pairs of points is $\frac{N(N-1)}{2}$.

Clearly $C(r)$ increases from 0 to 1 as r goes from 0 to ∞, and for a set of equally spaced points $C(r)$ will look much like a phase transition. Draw example.

The correlation dimension of the orbit S is d where $C(r) \approx r^d$ for sufficiently small r, i.e.

$$cordim(S) = \lim_{r\to 0} \frac{\log C(r)}{\log r}$$

Qualitatively, in higher dimensions, there are more ways for points to be close to each other. So the number of pairs close to each other will rise more rapidly in higher dimensions.

Show Fig 4.17.

This was created by iterating the Henon Map $N = 1000$ times. Of the $\frac{N(N-1)}{2} \approx 500,000$ possible pairs of points, the proportion within r of each other are counted for $r = 2^{-2}, 2^{-3}, \ldots, 2^{-8}$.

Lab visit 4: Measuring fractal dimension of an experimental attractor. Show Taylor-Couette flow in /fluids/ directory.

Show figure 4.19. Silicon oil fills annulus. Outer diameter 2 inches glass (fixed), inner 1 inch rotating at a rate controlled to within 10^{-4}s. h/d can be varied between 0 and 10, aspect of 0.374 used for this attractor.

Light scatters off of particles in the oil, velocity is measured at one point in physical space with a laser Doppler anemometer every 0.02 seconds.

The attractor is time-delay reconstructed by plotting vectors of the form $(y_t, y_{t+T}, y_{t+2T}, \cdots, y_{t+(m-1)T})$ in $\mathbb{R}^m$.

This technique approximates the attractor because quantities not measured are likely to depend somehow on velocity, but lagged at some time scale which the delay is designed to sample.

For the picture $T = 0.1s$ or 5 sampling units.

The embedding dimension, the number of degrees of freedom sufficient to describe properties of the state space via the time delay, is chosen to be $m = 2$ for this data.

Show figure 4.20a. Axes are velocity vs. velocity 0.1s later.

Show figure 4.20b. Axes are $\log_2(r)$ and the slope estimate returned for fitting a line to $[\log_2(C(r))$ vs $\log_2(r)]$ using all r values to the right. Messy business.

Curves for $m = 2, 3, \ldots, 20$ shown in figure 4.20b. The correlation dimension estimate is defined as the limit as $r \to 0$, this is $-\infty$ on the x-axis.

However, at the left end of the figure we see the impact of experimental noise, the straight line fit to $[\log_2(C(r))$ vs $\log_2(r)]$ degrades as $r \to 0$.

The range $2^{-5} < r < 2^{-2}$ shows agreement on a dimension of about 3.

This method can also be used to calculate the LE (model free) of data by measuring the rate of divergence of nearest neighbors and looking for consistency across various embedding dimensions. However, stochasticity can fool this method.

Show Lathrop 3 meter sphere video in /fluids/, liquid sodium explodes on contact with water.

Show fractal network structure of leaves in Pictures/science/fractals/leaves, benefits of self similarity and branching structure, slime mold Japan.

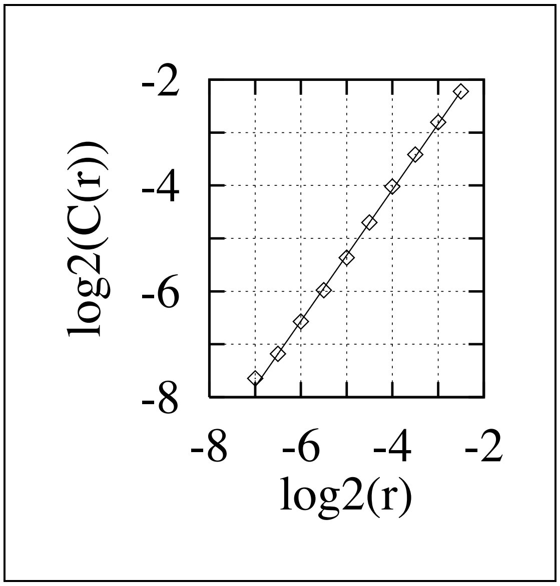


Figure 4.17 Correlation dimension of the Hénon attractor.
A graphical report of the results of the correlation dimension estimate for the Hénon attractor. The correlation dimension is the limit of $\log_2 C(r)/\log_2(r)$. The dimension corresponds to the limiting slope of the line shown, as $r \to 0$, which is toward the left. The line shown has slope ≈ 1.23.

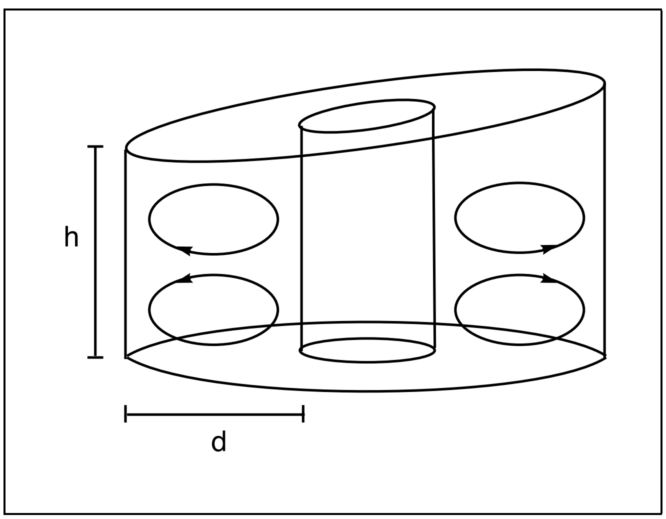


Figure 4.19 Setup of the Taylor-Couette flow experiment.
As the inner steel cylinder rotates, the viscous fluid between the cylinders undergoes complicated but organized motion.

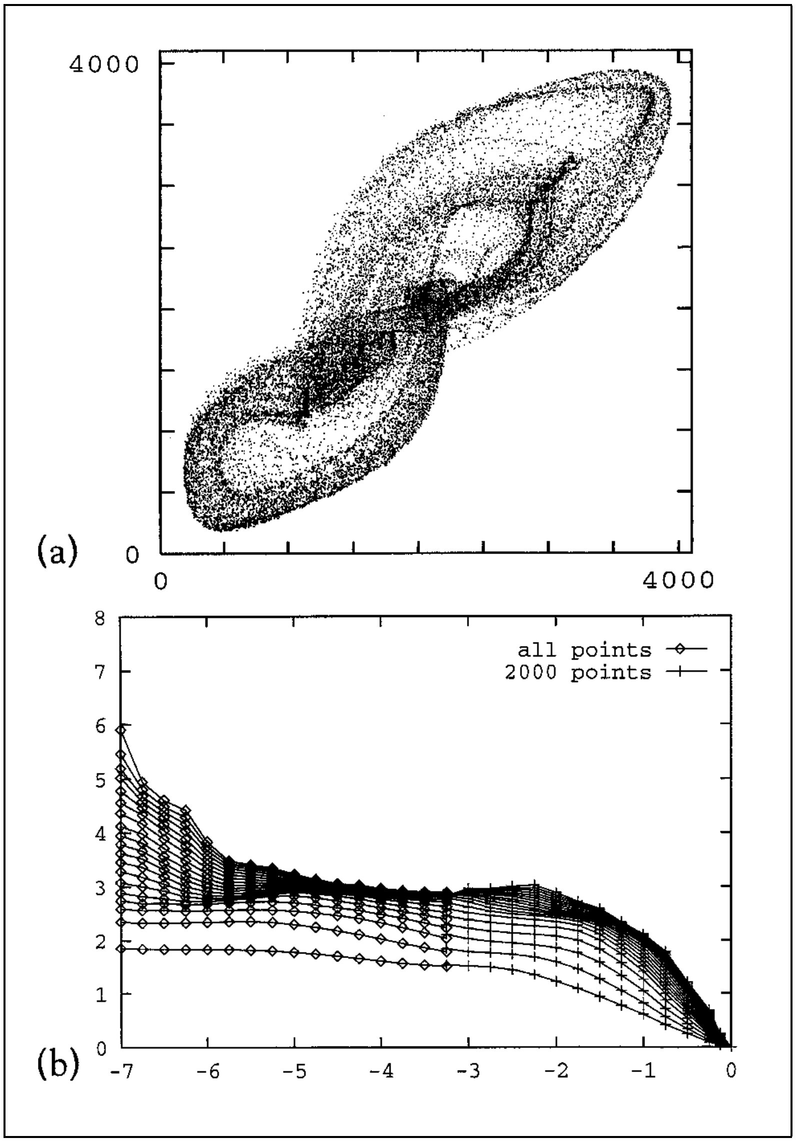


Figure 4.20 The Taylor-Couette data and its dimension.
(a) A two-dimensional projection of the reconstructed attractor obtained after filtering the experimental data. The time delay is $T = 5$. (b) Slope estimates of the correlation sum. There are 19 curves, each corresponding to correlation dimension estimates for an embedding dimension between 2 and 20. The 32,000 points plotted in Figure 4.20(a) were used in the left half, and only 2000 points in the right half, for comparison. The conclusion is that the correlation dimension is approximately 3.