One Dimensional Maps Chapter 1

4 lectures

presentations/2014/weather
presentations/2009/giv/climate

A dynamical system is a set of possible states, together with a deterministic rule for uniquely defining the present state in terms of past states. No stochasticity.

Two classes of dynamical systems:

- discrete-time: take current state as input, produce new state as output (e.g. numerical ODEs)
- continuous-time: limit as discrete system update time goes to zero (e.g. ODEs)

Define $f^k(x)$ to be the result of applying the function f to the initial state x a total of k times.

Given an initial condition (IC) x, we would like to characterize the system for large k.

ex) logistic growth, $g(x) = 2x(1-x)$, logistic.m, limiting behavior is $x = 0.5$, iterates get closer forever.

Cobweb Plots: vertical line to the function, horizontal line to $y = x$, repeat

logistic_plot.m, Draw function and line $y = x$. What are the fixed points of $g(x) = 2x(1-x)$?

Graphically it is the intersection of the function with $y = x$.

Algebraically it is the solution to the equation $x = 2x(1-x)$. Solve.

http://rocs.hu-berlin.de/D3/logistic/

http://www.complexity-explorables.org/flongs/logistic/

A function whose domain and range spaces are the same will be called a map.

Let x be a point and f be a map, then the orbit of x under f is the set of points $\{x, f(x), f^2(x), \ldots\}$.

A point p is said to be a fixed point of the map if $f(p) = p$.

Stability of Fixed Points: We want a fixed point to be unstable if nearby points move away (e.g. inverted pendulum).

The epsilon-neighborhood $N_\varepsilon(p)$ of a point p is the interval $\{x \in \mathbb{R}\ st.\ |x-p| < \varepsilon\}$. Draw number line, p, and $(p-\varepsilon, p+\varepsilon)$.

If $\exists\ \varepsilon > 0\ st\ \forall x \in N_\varepsilon(p)$

$$\lim_{k\to\infty} f^k(x) = p$$

then we call p a sink or an attracting fixed point.

If $\exists\ \varepsilon > 0$ st all $x \in N_\varepsilon(p)$ except p eventually map outside of $N_\varepsilon(p)$, then p is a source or a repelling fixed point.

Ex) $f(x) = \frac{1}{2}(3x - x^3)$. Compute fix points and stability on board. What is the basin of attraction of the $x = 1$ sink?

- $I_1 = (0, \sqrt{3})$ belongs to the basin ($\sqrt{3}$ is a root)
- $I_2 = (-2.148, -\sqrt{3})$ maps to I_1, (-2.148 is the point which maps to $\sqrt{3}$)
- $I_3 =$ small interval of points to the right of $x = 2.148$ which map to I_2
- $I_4 =$ small interval of points to the left of $x = 2.148$ which map to I_3
- $\cdots$

These intervals don't overlap, gaps fall in the basin of the $x = -1$ sink. Intervals are open, boundary points eventually map onto the fixed point source at the origin.

I_n get smaller as n increases, all lie between $\sqrt{5}$ and $-\sqrt{5}$. Since $f(\sqrt{5}) = -\sqrt{5}$ and $f(-\sqrt{5}) = \sqrt{5}$, neither is in the basin of either sink, draw period two orbit square.

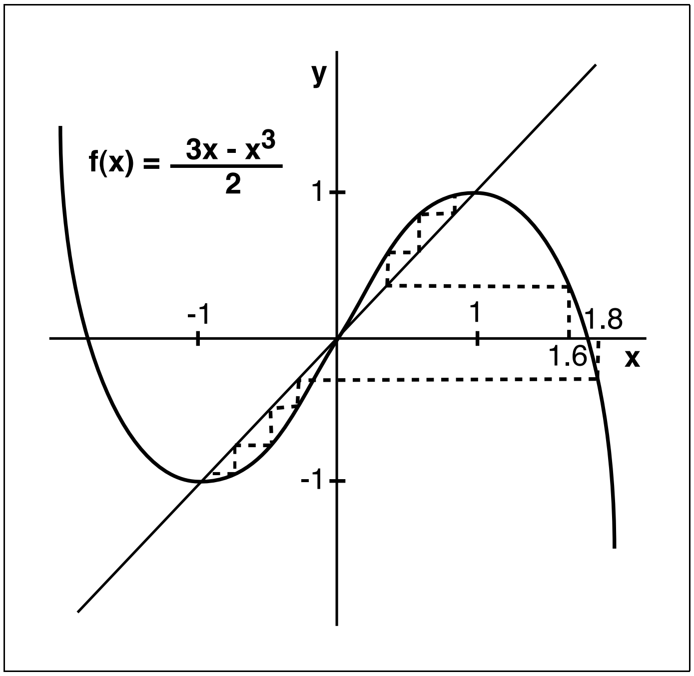


Figure 1.3 A cobweb plot for two orbits of $f(x) = (3x - x^3)/2$.
The orbit with initial value 1.6 converges to the sink at 1; the orbit with initial value 1.8 converges to the sink at -1.

Intercepts are at $\pm\sqrt{3}$.

Theorem: Let f be a smooth map on $\mathbb{R}$ (derivatives of all orders exist and are continuous) and assume p is a fixed point of f. If $|f'(p)| < 1$, then p is a sink. If $|f'(p)| > 1$, then p is a source. If $|f'(p)| = 1$, we need more information.

Let f be a map on $\mathbb{R}$. We call p a periodic point of period k if $f^k(p) = p$, and if k is the smallest such positive integer.

ex) $f(x) = -x$ on $\mathbb{R}$ has one fixed point, namely $x = 0$, all other points are period-2 since $f^2(x) = x$ is the identity.

Show table 1.2 and fig 1.4, period-2 orbits of $g(x) = 3.3x(1-x)$.

Let f be a map and assume p is a period-k point. The period-k orbit of p is a periodic sink (source) if p is a sink (source) for the map f^k.

Recall the chain rule: $(f \circ g)'(x) = f'\Big(g(x)\Big)g'(x)$ and if $f = g$, then $(f^2)'(x) = f'\Big(f(x)\Big)f'(x)$.

So if a is a period-2 point of f, e.g. $\{a, f(a)\}$, the chain rule says $(f^2)'$ is the same for both points on the orbit, i.e. $(f^2)'(a) = (f^2)'\Big(f(a)\Big)$.

ex) $g(x) = 3.3x(1-x)$ has a periodic orbit $\{.4794, .8236\}$, i.e.

$$(g^2)'(p_1) = g'(p_1)g'(p_2) = (g^2)'(p_2) = -0.2904$$

and $|-0.2904| < 1$ which implies the orbit is a sink.

Stability is a collective property of the periodic orbit, i.e. $(f^k)'(p_i) = (f^k)'(p_j) \;\; \forall \;\; i, j$.

The periodic orbit $\{p_1, p_2, ..., p_k\}$ is a sink (source) if $|f'(p_k)f'(p_{k-1}) \cdots f'(p_1)| < 1 \quad (> 1)$

Logistic Maps: $g_\mu(x) = \mu x(1-x)$

Loosely speaking, the parameter μ describes the reproduction rate. Show figure 1.5 and pull up LogisticCobwebChaos.gif

For $0 \leq \mu < 1$, the map has a sink at $x = 0$, i.e. small populations die.

$1 < \mu < 3$, the origin becomes unstable ($|g'_\mu(x)| = |\mu(1-2x)| > 1$)

map has sink at $x = \frac{\mu-1}{\mu}$, i.e. small populations grow to $\frac{\mu-1}{\mu}$. What type of bifurcation? (Period Doubling)

$\mu > 3$, the fixed point at $x = \frac{\mu-1}{\mu}$ becomes unstable and a period-2 sink develops, see `logistic_plot_g2.m`

This happens as a result of g^2 (degree 4) whose middle hump crosses $y = x$ from above when $\mu > 3$. For slightly larger μ, there are two intersections (p_1, p_2), a stable period-2 orbit.

$\mu > 1 + \sqrt{6} \approx 3.4495$, the period-2 sink becomes unstable and a period-4 sink is born.

To condense this information, that is the long term behavior of g for each μ, we use a bifurcation diagram. Show bifurcation diagram for the logistic map.

`https://www.desmos.com/calculator/3wldsysrhj`
`https://www.desmos.com/calculator/mqr1jsj7l9`

Steps to build a bifurcation diagram:

1. choose a parameter μ (x-axis)
2. choose a random seed $x \in (0,1)$
3. calculate orbit under $g_\mu(x)$
4. ignore transient states (e.g. first 1000)
5. plot the remaining orbit points
6. increment μ and repeat steps 2-5

Plotted points approximate fixed or periodic sinks, or other attracting sets. For example, $\mu = 3.4$ results in a period-2 sink, $\mu = 3.5$ results in a period-4 sink.

Note: period-2 sink still exists, but it is no longer stable so it doesn't appear in the bifurcation diagram.

Note: density variations in bifurcation diagram due to weakly unstable orbits, e.g. $\mu = 3.8282$ in matlab.

This sequence of periodic sinks, one for each $n = 1,2,3,...$ of period 2^n is a period-doubling cascade, a typical route to chaos in physical systems. If the nth period doubling occurs at $\mu = \mu_n$, then

$$\left\{\frac{\mu_2-\mu_1}{\mu_3-\mu_2}, \frac{\mu_3-\mu_2}{\mu_4-\mu_3}, \dots\right\}, \lim_{n\to\infty} \frac{\mu_{n+1}-\mu_n}{\mu_{n+2}-\mu_{n+1}} = 4.6692....$$

This number is universal for all 1 parameter maps, Feigenbaum's constant. For example, leaky faucet. See Lab Visit 10 for pictures.

For $\mu = 3.86$, the first few hundred iterates of an orbit appear to fill out much of the interval $[0,1]$.

Show fig 1.8, iterate in matlab `logistic_plot.m`. We also see periodic windows. Matlab $\mu = 3.83$ to see period 3 behavior.

Implications: Turn knob on climate model, e.g. importance of CO_2, can lead to drastically different behavior.

n	$g^n(x)$	$g^n(x)$	$g^n(x)$
0	0.2000	0.5000	0.9500
1	0.5280	0.8250	0.1568
2	0.8224	0.4764	0.4362
3	0.4820	0.8232	0.8116
4	0.8239	0.4804	0.5047
5	0.4787	0.8237	0.8249
6	0.8235	0.4792	0.4766
7	0.4796	0.8236	0.8232
8	0.8236	0.4795	0.4803
9	0.4794	0.8236	0.8237
10	0.8236	0.4794	0.4792
11	0.4794	0.8236	0.8236
12	0.8236	0.4794	0.4795
13	0.4794	0.8236	0.8236
14	0.8236	0.4794	0.4794

Table 1.2 Three different orbits of the logistic model $g(x) = 3.3x(1 - x)$.
Each approaches a period-2 orbit.

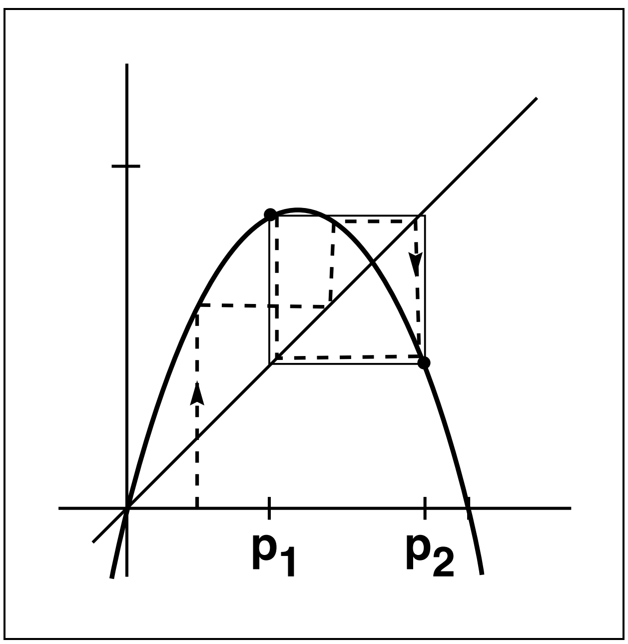


Figure 1.4 Orbit converging to a period-two sink.
The dashed lines form a cobweb plot showing an orbit which moves toward the sink orbit $\{p_1, p_2\}$.

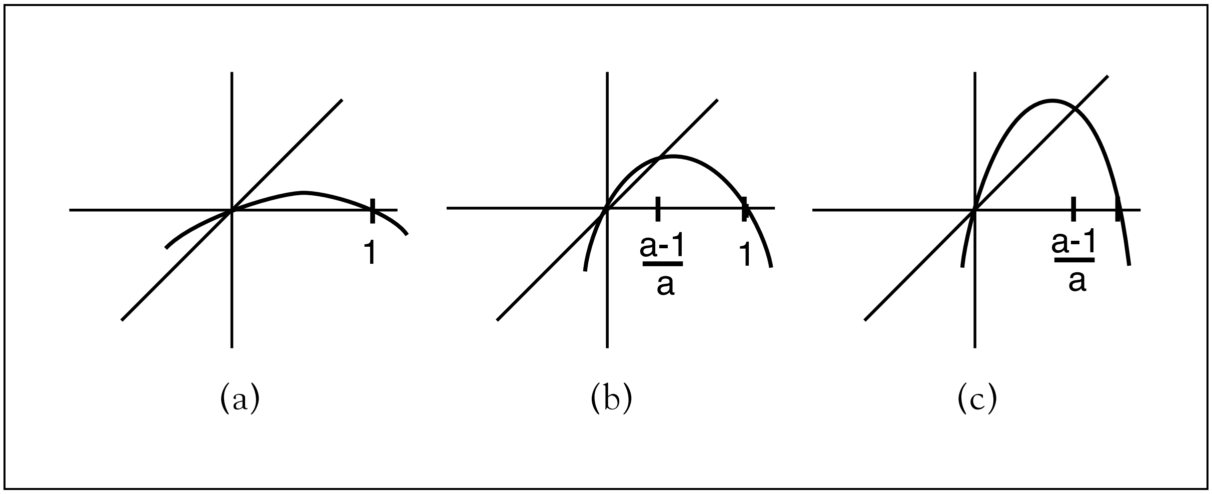


Figure 1.5 The logistic family.
(a) The origin attracts all initial conditions in [0, 1]. (b) The fixed point at $(a-1)/a$ attracts all initial conditions in $(0, 1)$. (c) The fixed point at $(a-1)/a$ is unstable.

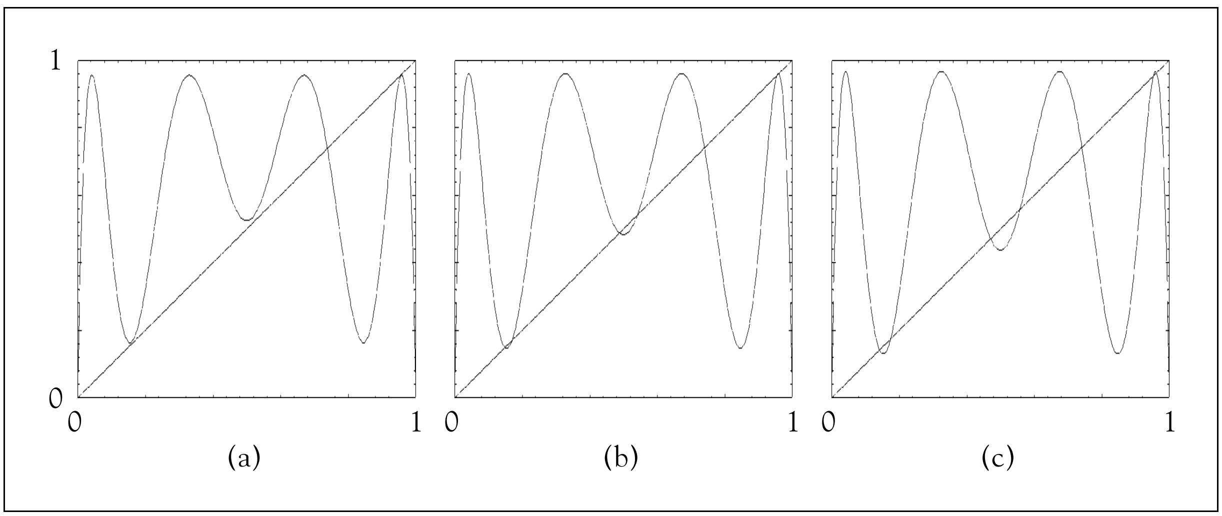
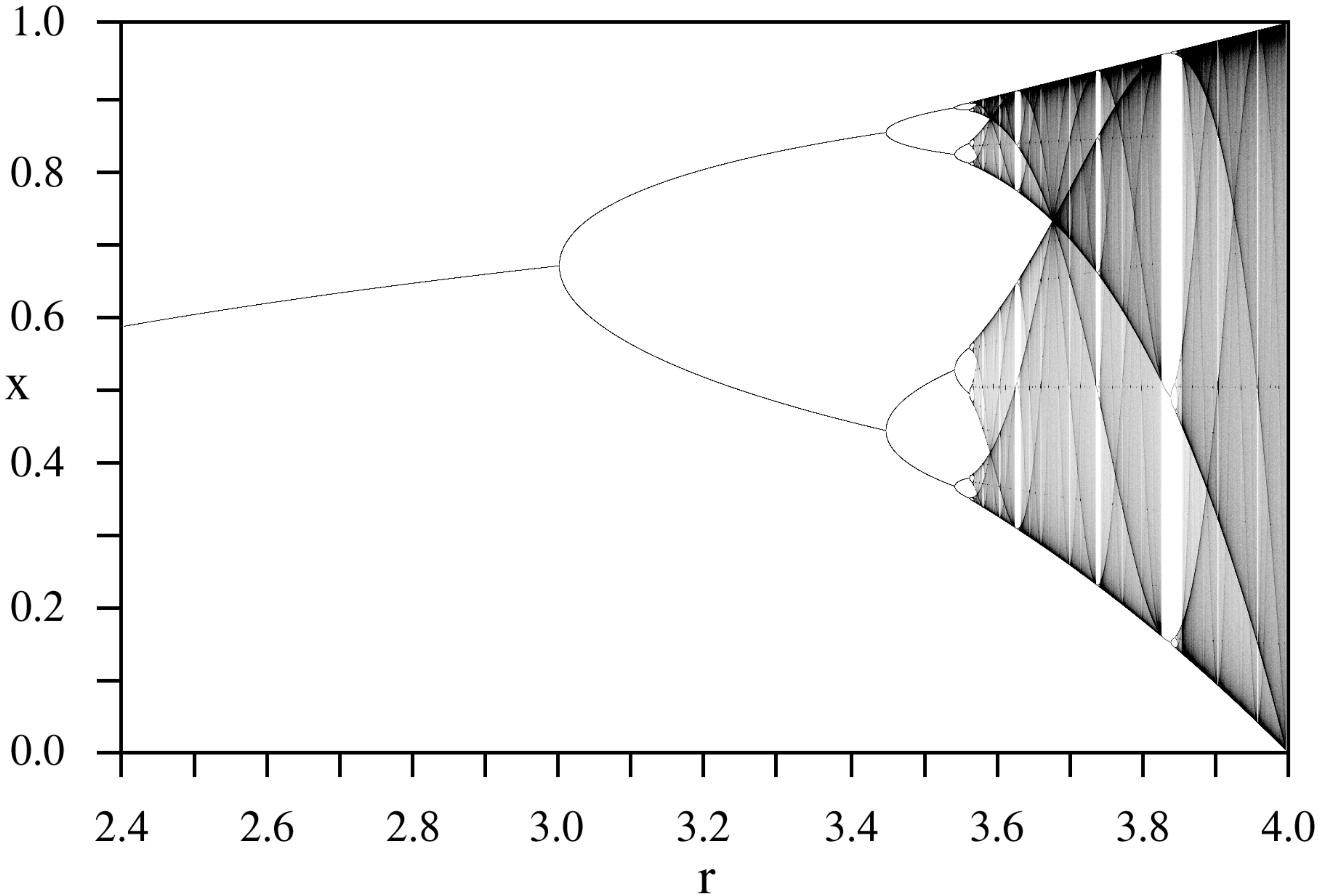


Figure 1.9 Graphs of the third iteration $g^3(x)$ of the logistic map $g_a(x) = ax(1-x)$.
Three different parameter values are shown: (a) $a = 3.82$ (b) $a = 3.84$ (c) $a = 3.86$.



Logistic map bifurcation diagram, $g_r = rx(1-x)$

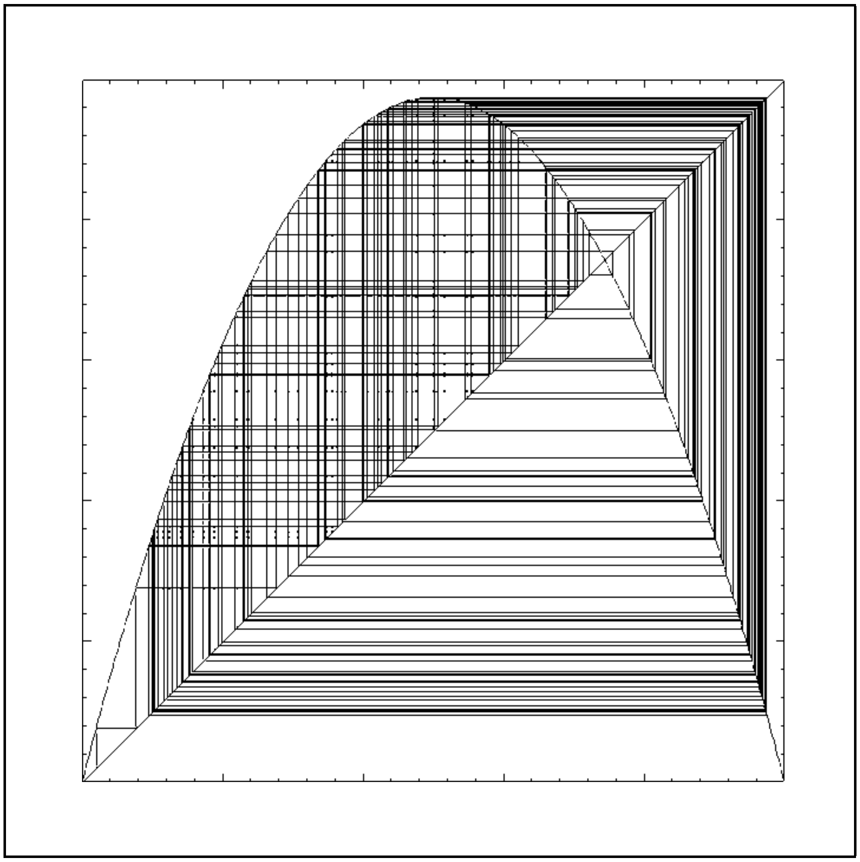


Figure 1.8 Cobweb plot for the logistic map.
A single orbit of the map $g(x) = 3.86x(1 - x)$ shows complicated behavior.

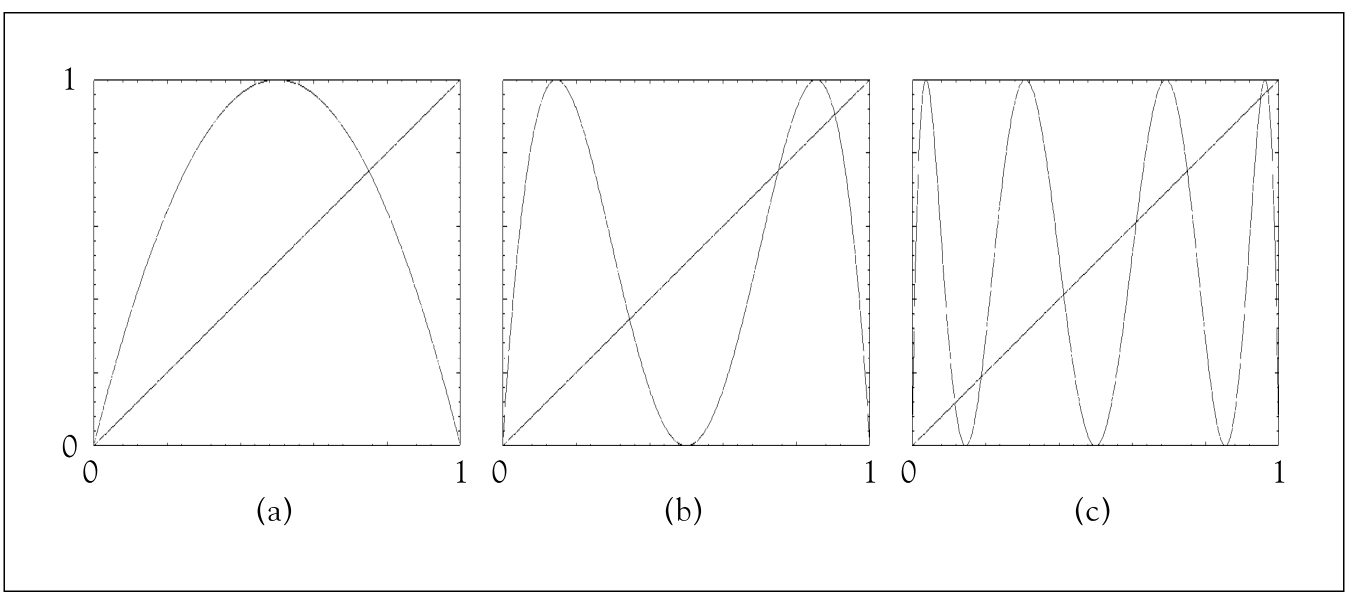


Figure 1.10 Graphs of compositions of the logistic map.
(a) the logistic map $G(x) = 4x(1 - x)$. (b) The map $G^2(x)$. (c) The map $G^3(x)$.

The logistic map with $a = 4$, namely $G(x) = 4x(1-x)$ is special (capital G). Show fig 1.10a.

The line $y = x$ intersects $G(x)$ at fixed points $x = 0, 3/4$. Both are unstable. Why?

Does G have any other periodic orbits? Hint: Any period-2 point of G will be a fixed point of G^2. Show figure 1.10b.

G^2 has four fixed points, two are fixed points of G, the other two p_1, p_2 make up a period-2 orbit, namely $G(p_1) = p_2$ and $G(p_2) = p_1$.

The map G^3 has eight fixed points, two of which are fixed points of G, namely $x = 0, 3/4$. Period-2 points of G are *not* fixed points of G^3. Why?

The remaining six fixed points of G^3 make up two period-3 orbits. Show example of period-3 orbit in matlab.

So... G^k has 2^k fixed points. This means there are infinitely many periodic orbits, at least one period-k orbit for every k, and they are ALL UNSTABLE!

Sensitive Dependence on Initial Conditions (SDIC): Butterfly effect.

y (mod 1) stands for the number $y+n$, where n is the unique integer ST $0 \leq y+n < 1$

e.g. 8.23 (mod 1) = 0.23 and -8.23 (mod 1) = 0.77.

Consider the map $f(x) = 3x \pmod 1$ on the unit interval (draw with empty circles at $(1/3,1), (2/3,1), (1,1)$), only 2 fixed points.

We call a point eventually periodic with period p for the map f if for some positive integer N, $f^{n+p}(x) = f^n(x)\ \forall n \geq N$, and if p is the smallest such positive integer

i.e. the orbit of x eventually maps onto a periodic orbit.

ex) $x = 1/3$ maps onto a period-1 orbit at the origin. $x = 2/27$? Rationals?

The $3x$ (mod 1) map magnifies any difference between nearby pairs of points by a factor of 3 at each iterate, see table 1.4.

Let f be a map on $\mathbb{R}$. A point x_0 has SDIC if $\exists d > 0$ ST any neighborhood of x_0 contains a point x ST $|f^k(x) - f^k(x_0)| \geq d$ for some nonnegative integer k.

i.e. A point x will be sensitive if it has neighbors as close as desired that eventually move away.

Itineraries: Assign the symbol L to the left subinterval $[0, 1/2]$ and R to the right subinterval $[1/2, 1]$.

Show figure 1.12 and draw L,R on y-axis, show where boundaries iterate to.

Fixed points: $x = 3/4$ for G, denoted $\overline{\text{R}}$.

$\{1/4, 3/4, 1/4, \cdots\}$ for 3x (mod 1) map is written $\overline{\text{LR}}$. The point 1/2 can be labelled either R or L.

What about itineraries for $2x(1-x)$?

What about itineraries for period-6 sink value of parameter found in logistic_plot?

Finish class with Pictures/science/fractals/ink-water-period-3.mp4

Spend the 4th class getting everyone caught up, Q+A, chain rule, matlab examples, introduce logistic_period for HW2.

Tweet: http://rocs.hu-berlin.de/D3/logistic/
Tweet: `https://juliadynamics.github.io/JuliaDynamics/`
Tweet: `https://www.youtube.com/watch?v=fDek6cYijxI`

n	$f^n(x_0)$	$f^n(y_0)$
0	0.25	0.2501
1	0.75	0.7503
2	0.25	0.2509
3	0.75	0.7527
4	0.25	0.2581
5	0.75	0.7743
6	0.25	0.3229
7	0.75	0.9687
8	0.25	0.9061
9	0.75	0.7183
10	0.25	0.1549

Table 1.4 Comparison of the orbits of two nearly equal initial conditions under the 3x mod 1 map.
The orbits become completely uncorrelated in fewer than 10 iterates.

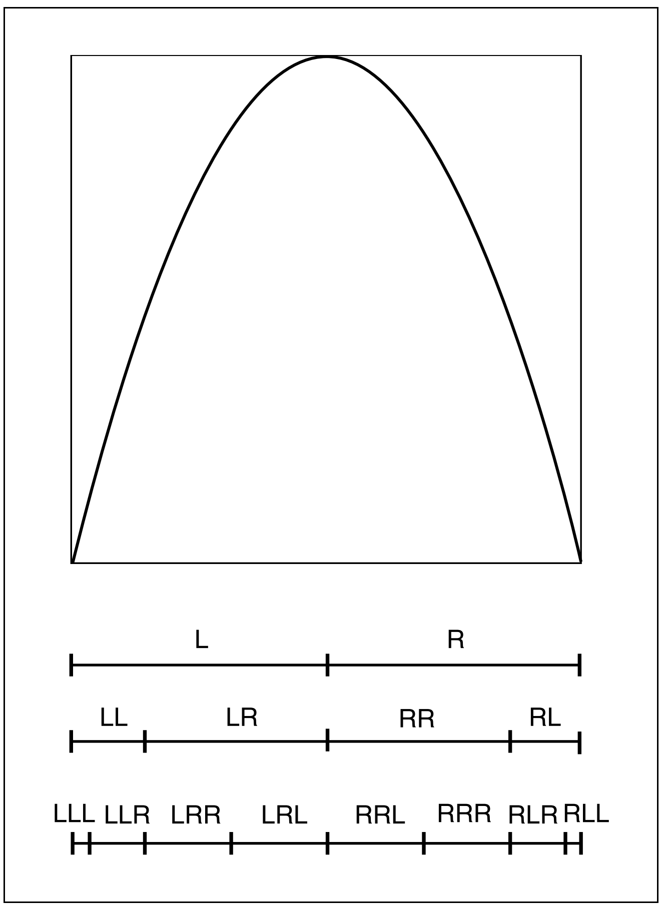
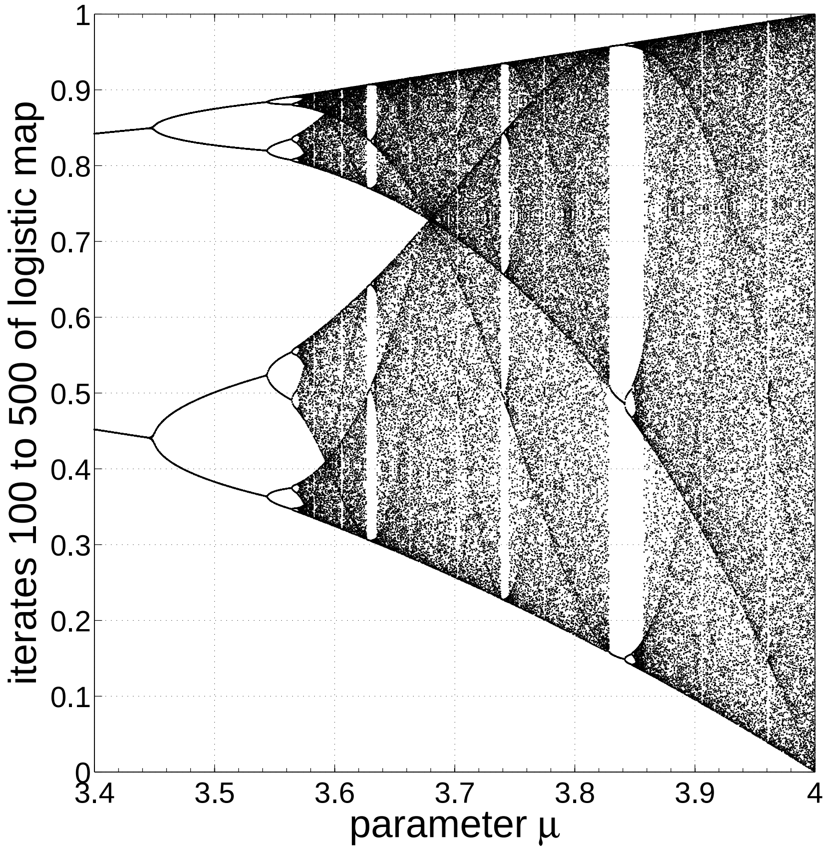
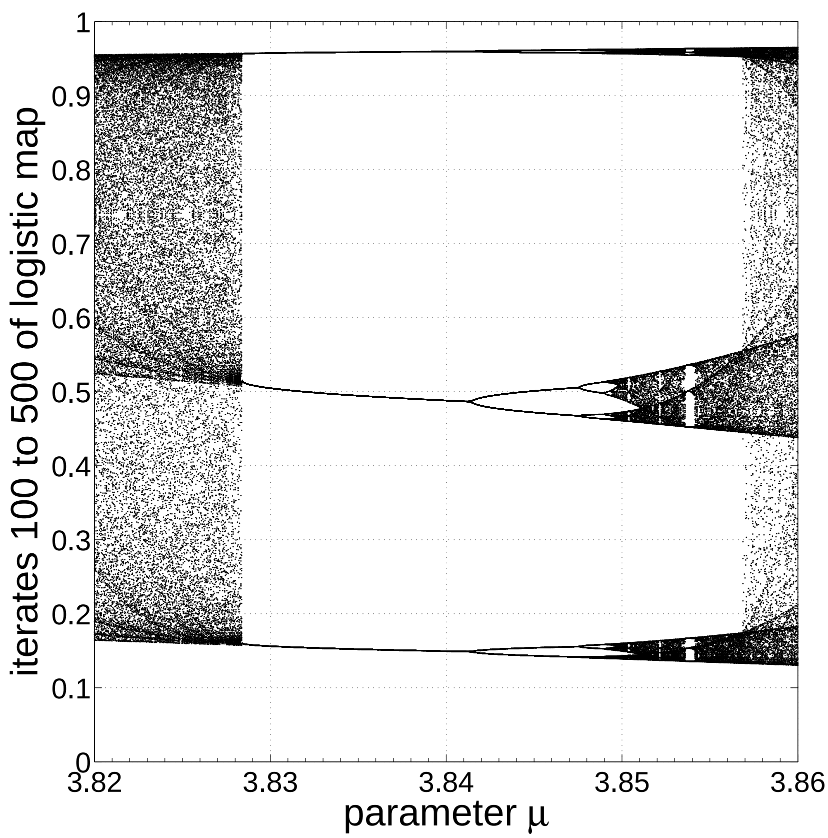


Figure 1.12 Schematic itineraries for $G(x) = 4x(1 - x)$.
The rules: (1) an interval ending in L splits into two subintervals ending in LL and LR if there is an even number of R's; the order is switched if there are an odd number of R's, (2) an interval ending in R splits into two subintervals ending in RL and RR if there are an even number of R's; the order is switched if there are an odd number of R's


iterates 100 to 500 of logistic map
parameter μ
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
3.4
3.5
3.6
3.7
3.8
3.9
4


iterates 100 to 500 of logistic map
parameter μ
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
3.82
3.83
3.84
3.85
3.86